1. **Problem**
A firm has the following production function:

$$F(K, L) = KL^2.$$

The price for one unit of *capital* is $p_K = 13$ and the price for one unit of *labor* is $p_L = 13$. Minimize the costs of the firm considering its production function and given a target production output of 430 units.

How high are in this case the minimal costs?

Solution
Step 1: Formulating the minimization problem.

$$\begin{aligned}\min_{K,L} C(K,L) &= p_K K + p_L L \\ &= 13K + 13L \\ \text{subject to:} \quad & F(K,L) = Q \\ & KL^2 = 430\end{aligned}$$

Step 2: Lagrange function.

$$\begin{aligned}\mathcal{L}(K, L, \lambda) &= C(K,L) - \lambda(F(K,L) - Q) \\ &= 13K + 13L - \lambda(KL^2 - 430)\end{aligned}$$

Step 3: First order conditions.

$$\frac{\partial \mathcal{L}}{\partial K} = 13 - \lambda L^2 = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial L} = 13 - 2\lambda K L^{2-1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(KL^2 - 430) = 0 \quad (3)$$

Step 4: Solve the system of equations for K, L, and λ.

Equating Equations (1) and (2) after solving for λ gives:

$$\begin{aligned}\frac{13}{L^2} &= \frac{13}{2KL^{2-1}} \\ K &= \frac{13}{2 \cdot 13} \cdot L^{2-(2-1)} \\ K &= \frac{13}{26} \cdot L\end{aligned}$$

Substituting this in the optimization constraint gives:

$$\begin{aligned}KL^2 &= 430 \\ \left(\frac{13}{26} \cdot L\right) L^2 &= 430 \\ \frac{13}{26} L^3 &= 430 \\ L &= \left(\frac{26}{13} \cdot 430\right)^{\frac{1}{3}} = 9.50968541 \ \approx \ 9.51 \\ K &= \frac{13}{26} \cdot L = 4.7548427 \ \approx \ 4.75\end{aligned}$$

The minimal costs can be obtained by substituting the optimal factor combination in the objective function:

$$
\begin{aligned}
C(K, L) &= 13K + 13L \\
&= 61.812955 + 123.62591 \\
&= 185.438865 \approx 185.44
\end{aligned}
$$

Given the target output, the minimal costs are 185.44.