1. **Problem**

A firm has the following production function:

$$F(K, L) = KL^2.$$

The price for one unit of *capital* is $p_K = 13$ and the price for one unit of *labor* is $p_L = 13$. Minimize the costs of the firm considering its production function and given a target production output of 430 units.

How high are in this case the minimal costs?

Solution

Step 1: Formulating the minimization problem.

$$\begin{aligned} \min_{K,L} C(K, L) &= p_K K + p_L L \\ &= 13K + 13L \\ \text{subject to:} F(K, L) &= Q \\ KL^2 &= 430 \end{aligned}$$

Step 2: Lagrange function.

$$\begin{aligned} \mathcal{L}(K, L, \lambda) &= C(K, L) - \lambda(F(K, L) - Q) \\ &= 13K + 13L - \lambda(KL^2 - 430) \end{aligned}$$

Step 3: First order conditions.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K} &= 13 - \lambda L^2 = 0 \\ \frac{\partial \mathcal{L}}{\partial L} &= 13 - 2\lambda K L^{2-1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -(KL^2 - 430) = 0 \end{aligned}$$

Step 4: Solve the system of equations for K, L, and λ.

Solving the first two equations for λ and equating them gives:

$$\begin{aligned} \frac{13}{L^2} &= \frac{13}{2KL^{2-1}} \\ K &= \frac{13}{2 \cdot 13} \cdot L^{2-(2-1)} \\ K &= \frac{13}{26} \cdot L \end{aligned}$$

Substituting this in the optimization constraint gives:

$$\begin{aligned} KL^2 &= 430 \\ \left(\frac{13}{26} \cdot L\right) L^2 &= 430 \\ \frac{13}{26} L^3 &= 430 \\ L &= \left(\frac{26}{13} \cdot 430\right)^{\frac{1}{3}} = 9.5096854 \approx 9.51 \\ K &= \frac{13}{26} \cdot L = 4.7548427 \approx 4.75 \end{aligned}$$

The minimal costs can be obtained by substituting the optimal factor combination in the objective function:

$$\begin{aligned} C(K, L) &= 13K + 13L \\ &= 61.812955 + 123.62591 \\ &= 185.438865 \approx 185.44 \end{aligned}$$

Given the target output, the minimal costs are 185.44.